1. (24 points) Consider the Kripke structure $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned}
W &= \{t, u, x, y\} \\
I(p) &= \{t, x\} \\
I(q) &= \{u, x, y\} \\
I(r) &= \{t, u\} \\
J(Mo) &= \{(t,u), (u,u), (u,t), (x,x), (y,y)\} \\
J(Pat) &= \{(t,t), (x,y), (y,t), (y,u)\} \\
J(Sam) &= \{(t,t), (t,y), (u,u), (x,u), (y,x), (y,t)\}
\end{aligned}$$

(a) (3 points) What is the value of $J(Sam\,|\,Pat)$?

$\{(t,t), (t,u), (y,y), (y,t)\}$

(b) (3 points) What is the value of $J(Mo\,|\,Pat)$?

$\{(u,t), (x,y), (y,t), (y,u)\}$

(c) (18 points) For each formula that follows, give the set of worlds in W in which it is true. (You do not need to show your work.)

i. $(p \wedge q) \vee r$

$\{x, t, u\}$

ii. $p \equiv r$

$\{t, y\}$

iii. Mo says $(\neg q)$

$\emptyset$

iv. Pat controls p

$\{t, x, y\}$

v. $Pat \Rightarrow Mo$

$\emptyset$

vi. $Mo \,\&\, Sam \Rightarrow Sam\,|\,Pat$

W

vii. $Mo\,|\,Pat$ says r

$\{u, t, y\}$

viii. Pat says $Pat \Rightarrow Mo$

$\{u\}$

ix. Sam says $(Pat$ says $Pat \Rightarrow Mo)$

$\{u, x\}$

2. (12 points) Give a formal proof of the following derivable rule:

$$\frac{P \text{ says } (R \text{ controls } \varphi) \qquad P\,|\,Q \text{ says } \varphi \qquad Q \Rightarrow R}{P \text{ says } \varphi}$$

1. P says $(R$ controls $\varphi)$	Given
2. $P \mid Q$ says φ	Given
3. $Q \Rightarrow R$	Given
4. $P \Rightarrow P$	Idempotency of $\Rightarrow$
5. $P \mid Q \Rightarrow P \mid R$	3,4 Monotonicity of $\Rightarrow$
6. $P \mid R$ says φ	2,5 Derived speaks for
7. P says R says φ	6, Quoting (and Equivalence)
8. P says $(R$ says $\varphi \supset \varphi)$	1, def controls
9. P says $(R$ says $\varphi \supset \varphi) \supset ((P$ says R says $\varphi) \supset P$ says $\varphi)$	MP Says
10. $(P$ says R says $\varphi) \supset P$ says φ	8,9 Modus ponens
11. P says φ	7,10 Modus ponens

3. (12 points) Prove that the following proposed inference rule is *sound*:

$$\frac{P \text{ controls } \varphi_1 \qquad \varphi_2}{P \text{ controls } (\varphi_1 \wedge \varphi_2)}$$

That is, show that—for all Kripke structures $\mathcal{M}$, principals P, and formulas $\varphi_1\varphi_2$—the following statement is true:

If $\mathcal{M} \models P$ controls φ_1 and $\mathcal{M} \models \varphi_2$, then $\mathcal{M} \models P$ controls $(\varphi_1 \wedge \varphi_2)$ as well.

PROOF: Consider an arbitrary Kripke structure $\mathcal{M} = \langle W, I, J \rangle$ such that

$$\mathcal{M} \models P \text{ controls } \varphi_1 \text{ and } \mathcal{M} \models \varphi_2.$$

We need to show that it's also the case that $\mathcal{M} \models P$ controls $(\varphi_1 \wedge \varphi_2)$.

By definition of $\mathcal{E}_\mathcal{M}$,

$$\mathcal{E}_\mathcal{M}[\![P \text{ controls } \varphi_1]\!] = (W - \mathcal{E}_\mathcal{M}[\![P \text{ says } \varphi_1]\!]) \cup \mathcal{E}_\mathcal{M}[\![\varphi_1]\!].$$

However, by definition of $\models$, we also know that $\mathcal{E}_\mathcal{M}[\![P \text{ controls } \varphi_1]\!] = W$. Likewise, because $\mathcal{M} \models \varphi_2$, we have that $\mathcal{E}_\mathcal{M}[\![\varphi_2]\!] = W$, and hence

$$\mathcal{E}_\mathcal{M}[\![\varphi_1 \wedge \varphi_2]\!] = \mathcal{E}_\mathcal{M}[\![\varphi_1]\!] \cap \mathcal{E}_\mathcal{M}[\![\varphi_2]\!] = \mathcal{E}_\mathcal{M}[\![\varphi_1]\!] \cap W = \mathcal{E}_\mathcal{M}[\![\varphi_1]\!].$$

By definition of $\mathcal{E}_\mathcal{M}$, we also have that

$$\begin{aligned}
\mathcal{E}_\mathcal{M}[\![P \text{ controls } (\varphi_1 \wedge \varphi_2)]\!] &= (W - \mathcal{E}_\mathcal{M}[\![P \text{ says } (\varphi_1 \wedge \varphi_2)]\!]) \cup \mathcal{E}_\mathcal{M}[\![\varphi_1 \wedge \varphi_2]\!] \\
&= (W - \{w \mid J(P)(w) \subseteq \mathcal{E}_\mathcal{M}[\![\varphi_1 \wedge \varphi_2]\!]\}) \cup \mathcal{E}_\mathcal{M}[\![\varphi_1 \wedge \varphi_2]\!] \\
&= (W - \{w \mid J(P)(w) \subseteq \mathcal{E}_\mathcal{M}[\![\varphi_1]\!]\}) \cup \mathcal{E}_\mathcal{M}[\![\varphi_1]\!] \\
&= (W - \mathcal{E}_\mathcal{M}[\![P \text{ says } \varphi_1]\!]) \cup \mathcal{E}_\mathcal{M}[\![\varphi_1]\!] \\
&= \mathcal{E}_\mathcal{M}[\![P \text{ controls } \varphi_1]\!] = W
\end{aligned}$$

It follows that $\mathcal{M} \models P$ controls $(\varphi_1 \wedge \varphi_2)$ as needed.

4. (12 points) Show that the following proposed inference rule is *not sound*, and therefore should not be added to the logic:

$$\frac{}{P \textsf{ controls } (Q \Rightarrow P)}$$

That is, give a particular Kripke structure $\mathcal{M}$ and principals P, Q such that:

$\mathcal{M} \not\models P \textsf{ controls } (Q \Rightarrow P)$.

For maximal credit, be sure to provide calculations and explanations to support your answer.

Consider the Kripke structure $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned} W &= \{a, b\} \\ J(P) &= \{(a, a)\} \\ J(Q) &= \{(a, b)\} \end{aligned}$$

Note that $J(P) \not\subseteq J(Q)$, and hence $\mathcal{E}_\mathcal{M}[\![Q \Rightarrow P]\!] = \emptyset$. We therefore have the following:

$$\begin{aligned} \mathcal{E}_\mathcal{M}[\![P \textsf{ says } Q \Rightarrow P]\!] &= \{w \mid J(P)w \subseteq \mathcal{E}_\mathcal{M}[\![Q \Rightarrow P]\!]\} \\ &= \{w \mid J(P)w \subseteq \emptyset\} \\ &= \{b\} \\ \mathcal{E}_\mathcal{M}[\![P \textsf{ controls } Q \Rightarrow P]\!] &= (W - \mathcal{E}_\mathcal{M}[\![P \textsf{ says } Q \Rightarrow P]\!]) \cup \mathcal{E}_\mathcal{M}[\![Q \Rightarrow P]\!] \\ &= (W - \{b\}) \cup \emptyset \\ &= W - \{b\} = \{a\} \neq W \end{aligned}$$

Thus, $\mathcal{M} \not\models P \textsf{ controls } Q \Rightarrow P$.