1. (24 points) Consider the Kripke structure $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned} W &= \{a, b, c, g\} \\ I(r) &= \{c, g\} \\ I(s) &= \{a, b, c\} \\ I(t) &= \{b, c, g\} \\ J(Sid) &= \{(a,a), (a,b), (b,c), (g,g)\} \\ J(Al) &= \{(a,b), (b,b), (c,b), (g,a)\} \\ J(Cam) &= \{(a,a), (a,b), (b,g), (b,c), (c,g), (g,g)\} \end{aligned}$$

(a) (2 points) What is the value of $J(Sid \mid Al)$?
$\{(a,b), (b,b), (g,a)\}$

(b) (2 points) What is the value of $J(Al \mid Cam)$?
$\{(a,c), (a,g), (b,c), (b,g), (c,c), (c,g), (g,a), (g,b)\}$

(c) (20 points) For each formula that follows, give the set of worlds in W in which it is true. You do not need to show your work.

i. $r \supset (s \wedge t)$
$\{a, b, c\}$

ii. $\neg(r \supset (s \wedge t))$
$\{g\}$

iii. Sid says r
$\{b, c, g\}$

iv. Sid controls r
$\{a, c, g\}$

v. Al says r
$\emptyset$

vi. Al controls r
W

vii. $Sid \Rightarrow Cam$
$\emptyset$

viii. $Al \mid Cam$ says r
$\{a, b, c\}$

ix. Sid says Al says r
$\{c\}$

x. $Cam \,\&\, Al \Rightarrow Sid \mid Al$
W

2. (12 points) Give a formal proof of the following derivable rule:

$$\frac{P \mid Q \text{ controls } (S \Rightarrow T) \qquad R \Rightarrow Q \qquad R \text{ says } (S \Rightarrow T)}{S \mid Q \Rightarrow T \mid Q}$$

In addition to the inference rules and definitions of Figure 3.1, you may use *without proof* any derived rule that appears in Chapter 3 of the text, including those listed in exercises.

Note: Do not use more than one logical rule on a single line of the proof.

1.	$P \mid Q$ controls $(S \Rightarrow T)$	Assumption
2.	$R \Rightarrow Q$	Assumption
3.	R says $(S \Rightarrow T)$	Assumption
4.	Q says $(S \Rightarrow T)$	2,3 Derived Speaks For
5.	P says Q says $(S \Rightarrow T)$	4, Says
6.	$(P \mid Q$ says $(S \Rightarrow T)) \equiv (P$ says Q says $(S \Rightarrow T))$	Quoting
7.	$P \mid Q$ says $(S \Rightarrow T)$	5,6 Equivalence
8.	$S \Rightarrow T$	1,7 Controls
9.	$Q \Rightarrow Q$	Idempotency of $\Rightarrow$
10.	$S \mid Q \Rightarrow T \mid Q$	8,9 Monotonicity of $\Rightarrow$

3. (12 points) Prove that the following proposed inference rule is *sound*:

$$\frac{P \Rightarrow Q \,\&\, R \qquad Q \text{ controls } \varphi}{P \text{ controls } \varphi}$$

That is, show that—for all Kripke structures $\mathcal{M}$, principals P, Q, R, and formulas φ—the following statement is true:

If $\mathcal{M} \models P \Rightarrow Q \,\&\, R$ and $\mathcal{M} \models Q$ controls φ, then $\mathcal{M} \models P$ controls φ as well.

PROOF: Consider an arbitrary Kripke structure $\mathcal{M} = \langle W, I, J \rangle$ such that

$$\mathcal{M} \models P \Rightarrow Q \,\&\, R, \qquad \mathcal{M} \models Q \text{ controls } \varphi.$$

By definition of $\models$,

$$\mathcal{E}_{\mathcal{M}}[\![P \Rightarrow Q \,\&\, R]\!] = W,$$

which means that $J(Q \,\&\, R) \subseteq J(P)$. Since $J(Q \,\&\, R) = J(Q) \cup J(R)$, it follows that $J(Q) \subseteq J(P)$.

Likewise, by definition of $\models$,

$$\begin{aligned} W &= \mathcal{E}_{\mathcal{M}}[\![Q \text{ controls } \varphi]\!] \\ &= (W - \mathcal{E}_{\mathcal{M}}[\![Q \text{ says } \varphi]\!]) \cup \mathcal{E}_{\mathcal{M}}[\![\varphi]\!], \end{aligned}$$

and hence $\mathcal{E}_{\mathcal{M}}[\![Q \text{ says } \varphi]\!]) \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]$.

Recall Fact 3 from lecture (15 Sept 2011):

> *For all principals P, Q, formulas ψ, and Kripke structures $\mathcal{M} = \langle W, I, J \rangle$, if $J(Q) \subseteq J(P)$, then $\mathcal{E}_{\mathcal{M}}[\![P \text{ says } \psi]\!] \subseteq \mathcal{E}_{\mathcal{M}}[\![Q \text{ says } \psi]\!]$.*

Thus, by Fact 3,

$$\mathcal{E}_{\mathcal{M}}[\![P \textsf{ says } \varphi]\!] \subseteq \mathcal{E}_{\mathcal{M}}[\![Q \textsf{ says } \varphi]\!] \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!],$$

and hence

$$\begin{aligned} W &= (W - \mathcal{E}_{\mathcal{M}}[\![P \textsf{ says } \varphi]\!]) \cup \mathcal{E}_{\mathcal{M}}[\![\varphi]\!], \\ &= \mathcal{E}_{\mathcal{M}}[\![P \textsf{ controls } \varphi]\!]. \end{aligned}$$

It follows that $\mathcal{M} \models P \textsf{ says } \varphi$. Because $\mathcal{M}$ was arbitrary, the rule is sound.

4. (12 points) Show that the following proposed inference rule is *not sound*, and therefore should not be added to the logic:

$$\frac{P \textsf{ says } \varphi_1}{\varphi_2 \supset P \textsf{ says } (\varphi_1 \wedge \varphi_2)}$$

That is, give a particular Kripke structure $\mathcal{M}$, formulas φ_1 and φ_2, and principal P such that:

$\mathcal{M} \models P \textsf{ says } \varphi_1$, but $\mathcal{M} \not\models \varphi_2 \supset P \textsf{ says } (\varphi_1 \wedge \varphi_2)$.

For maximal credit, be sure to provide calculations to support your answer.

ANSWER: Let φ_1 be the formula p and φ_2 be q; let P be the principal Al.

Consider the model $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned} W &= \{a, b\} \\ I(p) &= \{a\} \\ I(q) &= \{b\} \\ J(Al) &= \{(a, a), (b, a)\} \end{aligned}$$

Note that $\mathcal{E}_{\mathcal{M}}[\![p \wedge q]\!] = \mathcal{E}_{\mathcal{M}}[\![p]\!] \cap \mathcal{E}_{\mathcal{M}}[\![q]\!] = I(p) \cap I(q) = \emptyset$. In addition, we have

$$J(Al)a = \{a\}, \quad J(Al)b = \{a\}.$$

- $\mathcal{M} \models Al \textsf{ says } p$, because:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}[\![Al \textsf{ says } p]\!] &= \{w \mid J(Al)w \subseteq \mathcal{E}_{\mathcal{M}}[\![p]\!]\} \\ &= \{w \mid J(Al)w \subseteq \{a\}\} \\ &= \{a, b\} = W. \end{aligned}$$

- In contrast, $\mathcal{M} \not\models q \supset (Al \textsf{ says } (p \wedge q)$, because:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}[\![q \supset Al \textsf{ says } (p \wedge q)]\!] &= (W - \mathcal{E}_{\mathcal{M}}[\![q]\!]) \cup \mathcal{E}_{\mathcal{M}}[\![Al \textsf{ says } (p \wedge q)]\!] \\ &= (W - \{b\}) \cup \{w \mid J(Al)w \subseteq \mathcal{E}_{\mathcal{M}}[\![p \wedge q]\!]\} \\ &= (W - \{b\}) \cup \{w \mid J(Al)w \subseteq \emptyset\} \\ &= \{a\} \cup \emptyset \\ &= \{a\} \neq W \end{aligned}$$