An Algebraic Approach to the Constraint Satisfaction Problem

Matt Valeriote

McMaster University

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Outline

The Constraint Satisfaction Problem

Instance

A triple $P = (V, A, \mathcal{C})$ with

- V a nonempty, finite set of variables,
- A a nonempty, finite domain,
- $\mathcal{C}$ a set of constraints $\{C_1, \ldots, C_q\}$ where each C_i is a pair $(\vec{s}_i, R_i)$ with
  - $\vec{s}_i$ a tuple of variables of length m_i, called the scope of C_i, and
  - R_i a subset of A^{m_i}, called the constraint relation of C_i.

Question

Is there a solution to P, i.e., is there a function $f : V \to A$ such that for each $i \leq q$, the m_i-tuple $f(\vec{s}_i) \in R_i$?

The CSP

Scope of the CSP

Many combinatorial questions/problems can be expressed within the CSP framework:

- scheduling
- Artificial Intelligence (image processing)
- Graph Theory (colorability, unreachability)
- Logic (Boolean Satisfiability)
- Database Theory
- Computational Biology (protein folding)

Example (Graph q-colorability)

There is a natural translation of graph q-colorability problems into instances of the CSP. Consequently, the class of CSPs is NP-complete.

Alternate formulation

The homomorphism problem

Feder and Vardi point out that the class of CSPs is equivalent to the following class of problems:

- **Instance**: Two finite relational structures $\mathbf{A}$ and $\mathbf{B}$ of the same type.
- **Question**: Is there a homomorphism from $\mathbf{A}$ to $\mathbf{B}$?

Remark

Much of the literature on the CSP deals with it in terms of the homomorphism problem. For the algebraic approach it is often more natural to use the "variable-value" presentation.

Constraint Languages

Fact

As noted, the class of CSPs is NP-complete. By restricting the nature of the constraint relations that are allowed to appear in a CSP, it is possible to find natural subclasses of the CSP that are tractable.

Definition

Let A be a domain and Γ a set of finitary relations over A. CSP(Γ) is the collection of all instances of CSP with domain A and with constraint relations coming from Γ. Γ is called the constraint language of the class CSP(Γ).

The Complexity of Constraint Languages

Definition

Call a constraint language Γ tractable if the class of problems $\text{CSP}(\Gamma')$ is tractable for every finite subset Γ' of Γ. Call Γ NP-complete if $\text{CSP}(\Gamma')$ is NP-complete for some $\Gamma' \subseteq \Gamma$.

Big Problem:

Classify the tractable constraint languages.

Dichotomy Conjecture (Feder, Vardi):

Every constraint language is either tractable or NP-complete.

Some Constraint Languages

Example (Some Bad Constraint Languages)

- $\{\neq\}$ over some set of more than 2 elements. (graph colorability)
- The set of all finitary relations over $\{0, 1\}$. (Boolean Satisfiability)

Example (Some Good Constraint Languages)

- $\{=\}$ over some finite set.
- A set of hyperplanes of finite dimensional vector spaces over a finite field. (An instance of the CSP over this constraint language is essentially a system of linear equations and so Gaussian Elimination can be used to quickly solve it.)

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Polymorphisms

Definition

- An operation $f(x_1, x_2, \ldots, x_n)$ on a set A preserves the r-ary relation R on A if for all $\vec{s}_1, \ldots, \vec{s}_n \in R$:

$$(f(s_1^1, \ldots, s_n^1), \ldots, f(s_1^r, \ldots, s_n^r)) \in R$$

- In this case we say that f is a polymorphism of R and that R is an invariant relation of f.
- For Γ a set of relations, $Pol(\Gamma)$ denotes the set of functions that are polymorphisms of all relations in Γ.
- For F a set of functions, $Inv(F)$ denotes the set of all relations that are invariant under all operations from F.

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From constraint languages to algebras and back

Definition

An algebra $\mathbf{A}$ is a pair $\langle A, F \rangle$ where A is a nonempty set and F is a set of finitary operations on A.

The algebra of a constraint language

Let Γ be a constraint language over the set A. $\mathbf{A}_\Gamma$ denotes the algebra $\langle A, Pol(\Gamma) \rangle$.

The constraint language of an algebra

For $\mathbf{A} = \langle A, F \rangle$, $Inv(F)$ coincides with $\mathbf{SP}_{fin}(\mathbf{A})$, the subalgebras of finite powers of $\mathbf{A}$ (or the subpowers of $\mathbf{A}$). $\Gamma_\mathbf{A}$ denotes this constraint language.

Jeavon's Theorem

Note

Given a constraint language Γ over the set A, we can construct a larger constraint language as follows:

$$\Gamma \longrightarrow \mathbf{A}_\Gamma \longrightarrow \Gamma_{\mathbf{A}_\Gamma}.$$

Theorem (Jeavons)

Let Γ be a constraint language on A.

- *Γ is tractable if and only if $\Gamma_{\mathbf{A}_\Gamma}$ is.*
- *Γ is NP-complete if and only if $\Gamma_{\mathbf{A}_\Gamma}$ is.*

Corollary

To settle the Feder-Vardi Conjecture, it suffices to establish it for constraint languages of the form $\Gamma_{\mathbf{A}}$ for finite algebras $\mathbf{A}$.

Tractable Algebras

Definition

Call a finite algebra $\mathbf{A} = \langle A, F \rangle$ tractable (NP-complete) if the constraint language $\Gamma_{\mathbf{A}}$ $(= \mathbf{SP}_{fin}(\mathbf{A}))$ is tractable (NP-complete).

Note

The tractability of an algebra $\mathbf{A}$ is directly related to the properties of its subpowers, i.e., the set of subalgebras of $\mathbf{A}^n$, for $n > 0$.

Feder-Vardi Conjecture (Algebraic Version)

Every finite algebra is either tractable or NP-complete.

Some Tractable Algebras

Examples

Let **A** be a finite algebra. **A** is tractable if it is equipped with one of the following:

- A constant term.
- (Bulatov) A Mal'cev term: for all x, y,

$$p(y,x,x) = p(x,x,y) = y.$$

- A semilattice term: for all x, y,

$$x \wedge x = x, \quad x \wedge y = y \wedge x, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$$

- A near-unanimity term $t(\bar{x})$: for all x, y:

$$t(y,x,x,\ldots,x,x) = t(x,y,x,\ldots,x,x) = \cdots = t(x,x,\ldots,x,y) = x$$

Equations

Observation

The conditions on the preceding slide are expressed in terms of the existence of a term operation that satisfies a particular type of equation. These are special kinds of Mal'cev conditions.

Definition

A variety is a class of algebras defined by a set of equations.

Fact (Bulatov, Jeavons, Krohkin)

The tractability of a finite algebra only depends on its equational theory. So, if $\mathbf{A}$ *is tractable and* $\mathbf{B}$ *belongs to the variety determined by* $\mathbf{A}$ *then* $\mathbf{B}$ *is tractable.*

Varieties

Remarks

- Varieties are fundamental objects of study in universal algebra.
- Varieties can be classified according to the behaviour of the congruence lattices of its members.

Definition

Let **A** be an algebra.

- A congruence of **A** is an equivalence relation on A that is invariant under the operations of **A**. Equivalently, the congruences of **A** are the kernels of homomorphisms with domain **A**.
- The collection of the congruences of **A** forms a lattice under the ordering of inclusion, and is denoted $\mathrm{Con}\,\mathbf{A}$.

Special types of varieties

Definition

Let $\mathcal{V}$ be a variety.

- $\mathcal{V}$ is congruence distributive if the congruence lattice of every algebra in $\mathcal{V}$ satisfies the distributive law:

$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma).$$

- $\mathcal{V}$ is congruence permutable if for every $\mathbf{A} \in \mathcal{V}$ and congruences α, $\beta \in \mathrm{Con}\, \mathbf{A}$,

$$\alpha \circ \beta = \beta \circ \alpha.$$

- $\mathcal{V}$ is congruence modular if the congruence lattice of every algebra in $\mathcal{V}$ satisfies the modular law:

$$\alpha \leq \beta \Rightarrow \alpha \vee (\beta \wedge \gamma) = \beta \wedge (\alpha \vee \gamma).$$

Remarks

- Congruence conditions for varieties of the sort just mentioned can be characterized by the existence of special term operations. e.g., $\mathcal{V}$ is congruence permutable if and only if it has a term $p(x, y, z)$ that satisfies the equations $p(x, x, y) \approx p(y, x, x) \approx y$.
- A sometimes useful strategy for proving general results about varieties is to first establish the result in the congruence distributive and congruence permutable cases and then generalize to the congruence modular case.

The tractability conjecture

Theorem (Bulatov, Jeavons, Krokhin)

*To prove the Dichotomy Conjecture, it suffices to verify it for the constraint languages $\Gamma_{\mathbf{A}}$ for **A** a finite idempotent algebra. [An algebra is idempotent if each of its operations t satisfies the equation $t(x, x, \ldots, x) \approx x$.]*

Conjecture (Bulatov, Jeavons, Krokhin)

*A finite idempotent algebra **A** is tractable if and only if* var(**A**) *does not contain an algebra whose operations are all trivial (i.e., just projections). It is NP-complete otherwise.*

Note

Recall that var(**A**) denotes the variety generated by **A**.

Tame Congruence Theory

Tame Congruence Theory

Hobby and McKenzie have developed a notion of neighbourhood, or minimal set of a finite algebra. They show that the behaviour of minimal sets is limited to one of the following five types:

1. Unary
2. Affine
3. 2-element Boolean algebra
4. 2-element Lattice
5. 2-element Semi-lattice

Definition

- We say that a finite algebra $\mathbf{A}$ omits a particular type if no neighbourhoods of that type occur in $\mathbf{A}$.
- A variety $\mathcal{V}$ omits a particular type if each finite member of it does.

Neighbourhoods

Definition

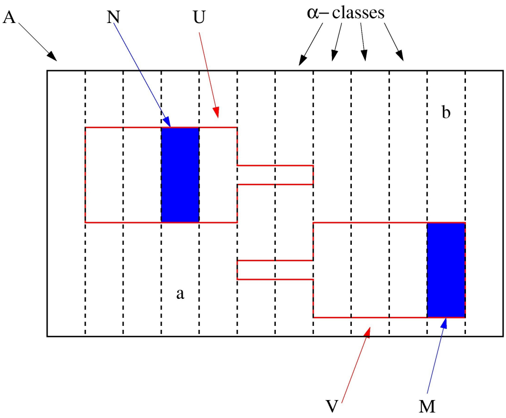
Let **A** be a finite algebra and α a minimal congruence of **A**.

- An α-minimal set of **A** is a subset U of A such that
  - $U = p(A)$ for some polynomial $p(x)$ of **A** that is not constant on the α-classes, and
  - U is minimal with this property.
- An α-neighbourhood of **A** is a subset N of A such that
  - $N = U \cap (a/\alpha)$ for some α-minimal set U and α-class (a/α), and
  - $|N| > 1$.

Facts

- *All α-minimal sets and neighbourhoods are isomorphic.*
- *The algebraic structure of each α-neighbourhood is of one of the five types from the previous slide.*

Neighbourhoods



Legend

- A is partitioned by the α-classes.
- U and V are α-minimal sets.
- $N = U \cap (a/\alpha)$ and $M = V \cap (b/\alpha)$ are α-neighbourhoods.

Definition

The type of α is equal to the type of any one of the α-neighbourhoods.

The tractability conjecture revisited

Remark

The tractability conjecture can be restated in terms of tame congruence theory as follows.

Conjecture

A finite idempotent algebra $\mathbf{A}$ *is tractable if and only if* $\mathrm{var}(\mathbf{A})$ *omits the unary type. It is NP-complete otherwise.*

Notes

- Bulatov, Jeavons and Krokhin have shown that if $\mathrm{var}(\mathbf{A})$ admits the unary type then $\mathbf{A}$ is NP-complete.
- [Bulatov] Given a finite idempotent algebra $\mathbf{A}$, the tractability condition from the conjecture can be tested in polynomial time. Given a finite constraint language Γ, testing for the condition (in the algebra $\mathbf{A}_\Gamma$) is an NP-complete problem.

Unanimity Operations

Definition

Let A be a set and $t(x_1, \ldots, x_n)$ an operation on A with $n > 2$.

- t is a weak near-unanimity operation if it is idempotent and satisfies the equations:

$$t(y, x, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, x, \ldots, x, y)$$

- t is a near-unanimity operation if it satisfies:

$$t(y, x, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, x, \ldots, x, y) = x$$

Fact

If **A** *has a near-unanimity term operation then it is tractable.*

Weak near unanimity terms

Examples

- Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice. Then the operation

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

  is a near unanimity operation.
- Let **G** be the group of integers, modulo n. Then the operation $x_1 + x_2 + \cdots + x_{n+1}$ is a weak near-unanimity operation.

Theorem (Maroti, McKenzie)

Let **A** *be a finite algebra. The following are equivalent:*

- var(**A**) *omits the unary type*
- **A** *has a weak near-unanimity term operation*
- *the congruence lattices of algebras in* var(**A**) *satisfy a particular (complicated) identity.*

Another formulation of the Tractability Conjecture

Conjecture

Let **A** *be a finite idempotent algebra. Then* **A** *is tractable if and only if* **A** *has a weak near-unanimity term operation.* **A** *is NP-complete otherwise.*

Remark

- Call a function $t(x_1, x_2, \ldots, x_n)$ on a set A cyclic if it satisfies the equation
$$t(x_1, x_2, \ldots, x_n) \approx t(x_2, x_3, \ldots, x_n, x_1).$$
- Recently it has been shown that if **A** is finite and idempotent then **A** has a cyclic term operation if var(**A**) is congruence distributive (Barto, Kozik, Niven) or if it is congruence permutable (Maroti, McKenzie).

Question

If var(**A**) omits the unary type must **A** have a cyclic term?

Tractability via local consistency

Definition

- Let Γ be a constraint language. We say that it has width k if whenever P is an instance of CSP(Γ) that is "locally k-consistent" then it has a solution. Γ has finite width if it has width k for some $k > 0$.
- A finite algebra **A** has width k (or finite width) if the constraint language Inv(**A**) does.

Fact

If Γ has finite width then it is globally tractable.

Note

- There are several closely related notions of width in the literature.
- The most prominent is one that is equivalent to a certain kind of definability within Datalog.

Finite width

Examples

A finite algebra with one of the following operations has finite width and hence is tractable:

- A semi-lattice operation $x \wedge y$, i.e., an operation that satisfies: $x \wedge x = x$, $x \wedge y = y \wedge x$, and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
- A near-unanimity operation.

Conjecture (Finite Width Conjecture)

Let $\mathbf{A}$ *be a finite idempotent algebra.* $\mathbf{A}$ *has finite width if and only if* $\mathrm{var}(\mathbf{A})$ *omits the unary and affine types.*

Note

Larose and Zadori have established one direction of this conjecture, namely that if $\mathbf{A}$ has finite width then $\mathrm{var}(\mathbf{A})$ omits the unary and affine types.

Omitting the unary and affine types

Theorem (Hobby, McKenzie)

Let **A** *be a finite algebra. The following are equivalent:*

- var(**A**) *omits the unary and affine types*
- *the congruence lattices of the algebras in* var(**A**) *are meet semi-distributive, i.e., satisfy the implication: if* $\alpha \wedge \beta = \alpha \wedge \gamma$ *then* $\alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma)$,
- *[McKenzie, 2006] For some* $N > 0$, **A** *has a* k*-variable weak near-unanimity term operation for all* $k > N$.

Notes

- The proof of McKenzie's 2006 result employs the meet semi-distributivity of the congruence lattices of members of var(**A**).
- Kiss-Valeriote show that finite width implies the existence of many weak near-unanimity term operations.

The finite width conjecture, revisited

Remark

Using McKenzie's 2006 result, the finite width conjecture can be restated as follows:

Conjecture (Finite Width Conjecture)

Let **A** *be a finite idempotent algebra. Then* **A** *has finite width if and only if for some* $N > 0$, **A** *has a* k*-variable weak near-unanimity term operation for all* $k > N$.

Facts

If the conjecture is true, it follows that

- *[Larose, Valeriote] there is a polynomial time algorithm to determine if a finite idempotent algebra has finite width.*
- *[Bulatov] the problem of determining if a finite constraint language has finite width is NP-complete.*

Congruence Distributivity

Theorem (Jónsson)

$\mathcal{V}$ is congruence distributive if and only if for some n there are terms $p_i(x, y, z)$ for $0 \leq i \leq n$ which satisfy the identities:

$$\begin{aligned}
p_0(x, y, z) &= x \\
p_n(x, y, z) &= z \\
p_i(x, y, x) &= x \text{ for all } i \\
p_i(x, x, y) &= p_{i+1}(x, x, y) \text{ for all } i \text{ even} \\
p_i(x, y, y) &= p_{i+1}(x, y, y) \text{ for all } i \text{ odd}
\end{aligned}$$

Definition

For $n > 1$, $CD(n)$ denotes the class of all algebras that have a sequence of $n + 1$ Jónsson terms.

Congruence Distributivity

Fact

The bounded width conjecture implies that all algebras that generate congruence distributive varieties have bounded width (since congruence distributive varieties omit the unary and affine types).

Theorem (Kiss, Valeriote)

Let $\mathbf{A}$ *be a finite algebra in* $CD(3)$ *of size* m*. Then* $\mathbf{A}$ *is tractable and:*

- *If* Γ *is a finite constraint language contained in* $\Gamma_{\mathbf{A}}$ *whose relations all have arity at most* k*, then* Γ *has relational width* k*.*
- *The (infinite) constraint language* $\Gamma_{\mathbf{A}}$ *has relational width* m^2*.*

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Tractability via small generating sets

Note

The tractability results for

- (Feder, Vardi) groups,
- (Jeavons, Cohen, Cooper) near unanimity operations,
- (Bulatov) Mal'cev operations, and
- (Dalmau) generalized majority-minority operations

can all be understood using the following algebraic property:

Definition

A finite algebra $\mathbf{A}$ has few subpowers if there is some polynomial $p(n)$ such that for each $n > 0$,

$$\log_2 |\{B \,:\, B \text{ is a subuniverse of } \mathbf{A}^n\}| \leq p(n).$$

Algebras with few subpowers

Note

Algebras with few subpowers have been studied in various guises by Bulatov, Chen, and Dalmau. Chen and Dalmau both conjectured that if $\mathbf{A}$ is a finite algebra having few subpowers then $\mathbf{A}$ is tractable.

Theorem (Idziak, Markovic, McKenzie, Valeriote, Willard)

If $\mathbf{A}$ is a finite algebra having few subpowers then $\mathbf{A}$ is globally tractable.

Note

The proof modifies Dalmau's gmm algorithm and uses the fact that if an algebra has few subpowers then all of its subpowers have small generating sets.

A characterization of algebras with few subpowers

Definition

A k-edge operation on a set A is a $k+1$-variable operation t that satisfies the equations:

$$\begin{aligned} t(x,x,y,\dots,y) &= t(x,y,x,y,\dots,y) = y, \\ t(y,y,y,x,y,\dots,y) &= t(y,y,y,y,x,y,\dots,y) = \cdots \\ \cdots &= t(y,y,y,\dots,y,x) = y. \end{aligned}$$

Theorem (Berman, Idziak, Markovic, McKenzie, Valeriote, Willard)

Let $\mathbf{A}$ *be a finite algebra. Then* $\mathbf{A}$ *has few subpowers if and only if it has a* k*-edge term operation for some* $k > 1$*.*

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Concluding remarks

Summary

- Several results and open problems that deal with the CSP can be recast in purely algebraic terms.
- From the algebraic point of view, settling the CSP conjectures will involve investigating the interplay between sufficiently strong terms on an algebra $\mathbf{A}$ and the behaviour of the subalgebras of $\mathbf{A}^n$, $n > 0$.
- The classification of varieties according to congruence identities and tame congruence theory provides an approach to settling or at least partially verifying the CSP conjectures.
- Work on the CSP has led to the discovery of novel properties of algebras that are of interest independent of the CSP.