Linear Programming

DPV Chapter 7, Part 2

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Part 2

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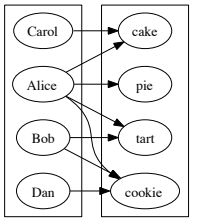
Uncredited diagrams are from DPV or homemade.

Bipartite matching, 1

Definition

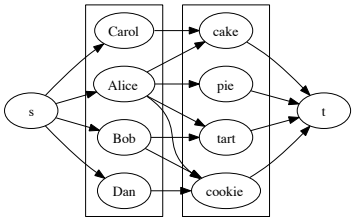
(a) A graph $G = (V, E)$ is *bipartite* when V can be partitioned into two nonempty sets L and R and each edge in E has one endpoint in L and the other in R.

(b) Given a bipartite graph G, a *bipartite matching* is a set of edges which give an 1-1 correspondence between L and R. *(E.g., {(Alice,pie), (Bob,tart), (Carol,cake), (Dan,cookie)})*



We can turn the problem of finding a bipartite matching into a network flow problem by:

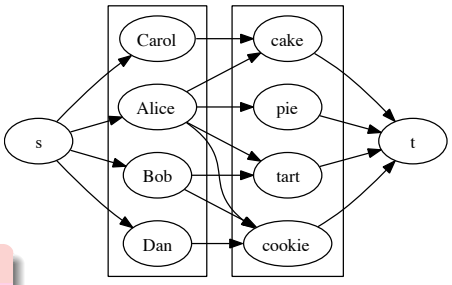
- Adding a source and target verts s and t.
- Adding edges from s to each $u \in L$.
- Adding edges from each $v \in R$ to t.
- Setting the capacity of each edge 1.



Bipartite matching, 2

bipartite matching $\rightarrow$ network flow

- Add a source and target verts s and t.
- Add edges from s to each $u \in L$.
- Add edges from each $v \in R$ to t.
- Set the capacity of each edge 1.



Problem

We do **not** want fractional flows.
(E.g., Dan wants a whole cookie!)

Saving Fact

If all edge capacities are integers, then our algorithm finds an integral optimal flow.

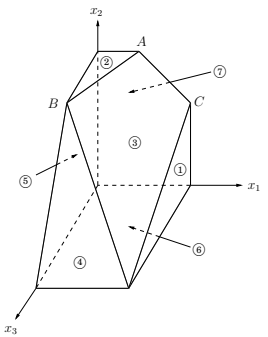
BUT!

In general it is hard to find solutions to LP problems if you demand all variables have integer values.
(Set up for Chapter 8.)

Simplex at a very high level, 1

$v \leftarrow$ any vertex in the feasible region
while there is a neighbor v' of v with a better objective value **do** $v \leftarrow v'$

- With n variables, we are working in $\mathbb{R}^n$.
- The feasible region is defined by the constraints:
  - $a_1x_1 + \cdots + a_nx_n = b$ defines a hyperplane in $\mathbb{R}^n$
  - $a_1x_1 + \cdots + a_nx_n \leq b$ and $a_1x_1 + \cdots + a_nx_n \geq b$ define halfspaces in $\mathbb{R}^n$.
  - The intersection of halfspaces gives a convex polyhedron.



$$\begin{aligned} \max \quad & x_1 + 6x_2 + 13x_3 & \\ & x_1 \leq 200 & ① \\ & x_2 \leq 300 & ② \\ & x_1 + x_2 + x_3 \leq 400 & ③ \\ & x_2 + 3x_3 \leq 600 & ④ \\ & x_1 \geq 0 & ⑤ \\ & x_2 \geq 0 & ⑥ \\ & x_3 \geq 0 & ⑦ \end{aligned}$$

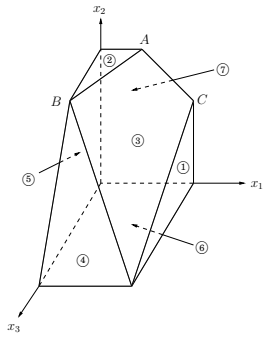
Simplex at a very high level, 2

$v \leftarrow$ any vertex in the feasible region
while there is a neighbor v' of v with a better objective value **do** $v \leftarrow v'$

Q: What is a vertex? A: A "corner."

- In 2-D, two lines intersect in a point*.
- In 3-D, three planes intersect in a point*.
  E.g., ①, ②, and ⑦ with equality meet at point A.
- In n-D, n hyperplanes intersect in a point*.

*Watch out for degeneracy!



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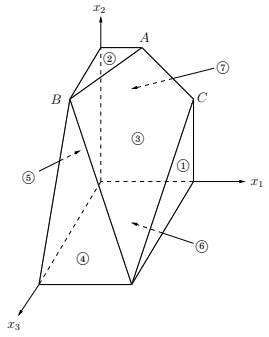
Simplex at a very high level, 3

$v \leftarrow$ any vertex in the feasible region
while there is a neighbor v' of v with a better objective value **do** $v \leftarrow v'$

Q: What is a neighbor of a vertex?

A: They have $n-1$ defining constraints in common.
E.g., A and C share ③ and ⑦.



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Simplex: *Towards the Algorithm*

The algorithm has two basic jobs

- job 1: See if the current vertex is optimal. (If so, halt.)
- job 2: Determine where to move next.

Both jobs are easy at the origin: $\vec{0} = (0, \ldots, 0)$.

Why is the origin so special?

$\max: \vec{c}^T\vec{x}$
$A\vec{x} \le \vec{b}$
$\vec{x} \ge \vec{0}$

Suppose $\vec{0}$ is feasible.

- job 1: $\vec{0}$ is optimal $\iff$ $c_i \le 0$ for each i. *(Why?)*
- job 2: If $\vec{0}$ is not optimal, pick some i with $c_i > 0$ and increase x_i until we hit another constraint.

What do we do if we are not at the origin?

Transform coordinates to move the origin to where we are.

Simplex: *Moving the origin*

- Suppose we want to move the origin to $\vec{u}$.
- Suppose $\vec{u}$ is given by the constraints: $\vec{a}_i \cdot \vec{x} \le b_i$ where $i = 1, \ldots, n$.
- For $i = 1, \ldots, n$, let $y_i = b_i - \vec{a}_i \cdot \vec{x}$. Then:

in $\vec{x}$-space		in $\vec{y}$-space
$\vec{a}_i \cdot \vec{x} \le b_i$	$\rightsquigarrow$	$y_i \ge 0$
$\vec{u}$	$\rightsquigarrow$	$\vec{0}$
max: $\vec{c}^T \cdot \vec{x}$	$\rightsquigarrow$	max: $(c_u + \vec{c'}^T \cdot \vec{y})$

where c_u = the cost at $\vec{u}$ and
$\vec{c'}$ = the transformed cost vector

Example from DPV, 1

Initial LP:

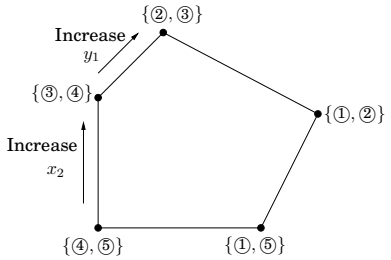
$$\begin{array}{rcll} \max\ 2x_1 + 5x_2 & & & \\ 2x_1 - x_2 & \leq & 4 & ① \\ x_1 + 2x_2 & \leq & 9 & ② \\ -x_1 + x_2 & \leq & 3 & ③ \\ x_1 & \geq & 0 & ④ \\ x_2 & \geq & 0 & ⑤ \end{array}$$

Current vertex: $\{④, ⑤\}$ (origin).
Objective value: 0.

Move: increase x_2.
⑤ is released, ③ becomes tight. Stop at $x_2 = 3$.

New vertex $\{④, ③\}$ has local coordinates (y_1, y_2):

$$y_1 = x_1, \quad y_2 = 3 + x_1 - x_2$$



Example from DPV, 2

Rewritten LP:

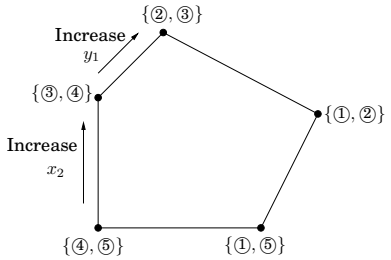
$$\begin{array}{rcll} \max\ 15 + 7y_1 - 5y_2 & & & \\ y_1 + y_2 & \leq & 7 & ① \\ 3y_1 - 2y_2 & \leq & 3 & ② \\ y_2 & \geq & 0 & ③ \\ y_1 & \geq & 0 & ④ \\ -y_1 + y_2 & \leq & 3 & ⑤ \end{array}$$

Current vertex: $\{④, ③\}$.
Objective value: 15.

Move: increase y_1.
④ is released, ② becomes tight. Stop at $y_1 = 1$.

New vertex $\{②, ③\}$ has local coordinates (z_1, z_2):

$$z_1 = 3 - 3y_1 + 2y_2, \quad z_2 = y_2$$



Example from DPV, 3

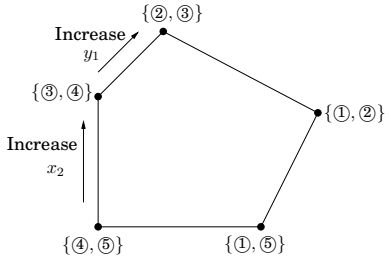
Rewritten LP:

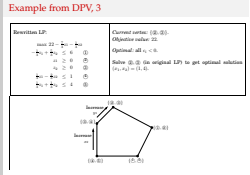
$$\begin{array}{rcll} \max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 & & & \\ -\frac{1}{3}z_1 + \frac{5}{3}z_2 & \leq & 6 & ① \\ z_1 & \geq & 0 & ② \\ z_2 & \geq & 0 & ③ \\ \frac{1}{3}z_1 - \frac{2}{3}z_2 & \leq & 1 & ④ \\ \frac{1}{3}z_1 + \frac{1}{3}z_2 & \leq & 4 & ⑤ \end{array}$$

Current vertex: $\{②, ③\}$.
Objective value: 22.

Optimal: all $c_i < 0$.

Solve ②, ③ (in original LP) to get optimal solution $(x_1, x_2) = (1, 4)$.





$(x_1, x_2) = (1, 4)$

Simplex Issues: *Finding the starting vertex*

- The origin is not always in the feasible region.

Q: So how to find a starting vertex?

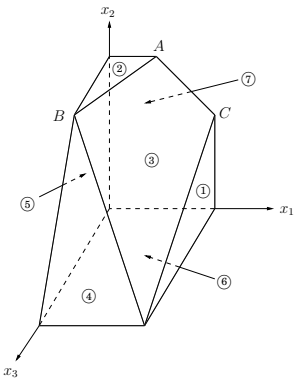
A: It turns out this is another LP problem:
Suppose we have $\min : \vec{c}^T \cdot \vec{x} \quad \ni \quad A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$.
Then:

- Make sure each $b_i \geq 0$.
  If $b_i < 0$, multiply both sides of the i-th eqn. by -1.
- Create new variables $z_1, \ldots, z_m \geq 0$ where m = # of equations.
- Add z_i to the left-hand side of equation i.
- $\min : z_1 + \cdots + z_m$ is the objective function.
- $z_1 = b_1, \ldots, z_m = b_m$ and all other variables = 0 is a starting vertex.

Case: Simplex reports 0 is the optimal value of the objective function.
Then ignore the z_i's and the value of the other vars gives a starting vertex.

Case: Simplex reports 0 is **not** the opt. value of the objective function.
Then the original LP problem must be infeasible. *(Why?)*

Simplex Issues: *Degeneracy*



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Vertex B is degenerate in that it is given by any one of:

- ②, ③, ④
- ②, ③, ⑤
- ②, ④, ⑤
- ③, ④, ⑤

Simplex Issues: *Degeneracy & Unboundedness*

- Degeneracy can confuse poor Simplex —
  causing it to return suboptimal answers.
- One way out is to perturb the b_i's by a small amount,
  i.e., $b_i \mapsto b_i \pm \epsilon_i$.
- This jolts each plan a bit and splits degenerate vertices into multiple verts.

Unboundedness

- Simplex can detect if the feasible region is unbounded
  in which case it halts and fusses.

Simplex Running Time

- With some cleverness (see DPV) each iteration can be gotten down to $O(m \cdot n)$ time, where

$$m = \text{the number of inequalities}$$
$$n = \text{number of variables}$$

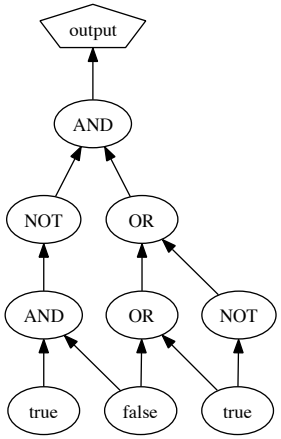
- There are $\binom{m+n}{n}$ many vertices, which is exponential size.
- There are cases where Simplex takes exponential time!!
- In practice, Simplex usually runs much faster than any other LP algorithm — including Karmarkar's poly-time LP algorithm.

Boolean Circuit Evaluation

Boolean Circuit

A directed, acyclic graph of gates of the following sorts:

- inputs gates: indegree 0, labeled `True` or `False`
- AND gates, OR gates: indegree 2
- NOT gates: indegree 1
- One of the gates is chosen as the *output gate*



Boolean Circuit Evaluation as an LP problem

- For each gate, create a variable x_g constrained:
  $0 \leq x_g \leq 1$.
- If g is a True-input gate, add:
  $x_g = 1$.
- If g is a False-input gate, add:
  $x_g = 0$.
- If g is an Or-gate with inputs from h and h', add:
  $x_g \geq x_h, \quad x_g \geq x_{h'}, \quad x_g \leq x_h + x_{h'}$
- If g is an And-gate with inputs from h and h', add:
  $x_g \leq x_h, \quad x_g \leq x_{h'}, \quad x_g \geq x_h + x_{h'} - 1$
- If g is a Not-gate with input from h, add:
  $x_g = 1 - x_h$

Boolean Circuit Evaluation as an LP problem—So?

- Boolean Circuit Evaluation is, in a certain sense (see Chapter 8), the most general problem solvable in polytime.

  I.e., every polytime problem reduces to a Boolean Circuit Evaluation.
- Boolean Circuit Evaluation reduces to LP.

$\therefore$ Every polytime problem reduces to LP!!

- Similarly, every polytime problem reduces to a Dynamic Programming problem.