Dynamic Programming

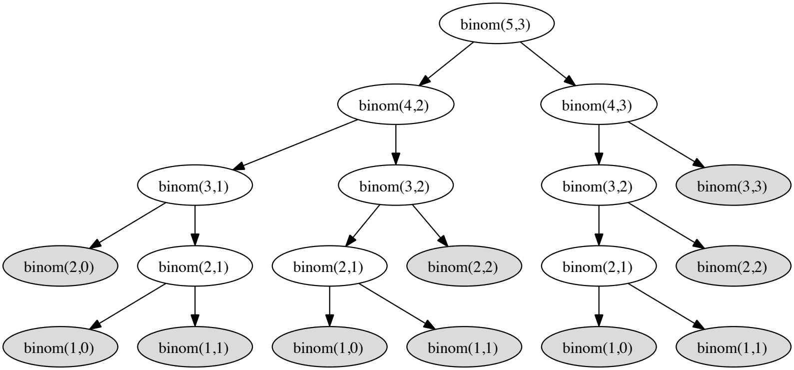
DPV Chapter 6, Part 1

Jim Royer

March 6, 2019

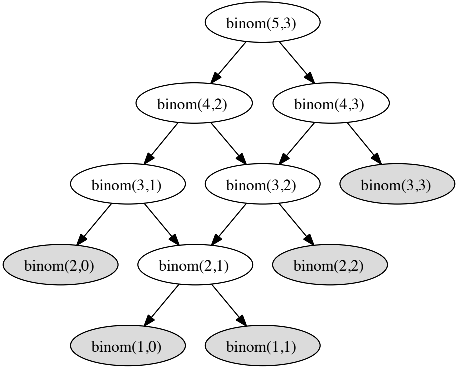
Computing binomial coeffecients, 1

$$binom(n,k) = \begin{cases} 1, & \text{if } k = 0 \text{ or } k = n; \\ binom(n-1,k-1) + binom(n-1,k), & \text{otherwise.} \end{cases}$$



Computing binomial coeffecients, 2

$$binom(n,k) = \begin{cases} 1, & \text{if } k = 0 \text{ or } k = n; \\ binom(n-1,k-1) + binom(n-1,k), & \text{otherwise.} \end{cases}$$



Computing binomial coeffecients, 3

Before Memoization

function binom(n, k)
  if $k = 0$ or $k = n$ **then return** 1
  else return binom$(n-1, k-1)$ + binom$(n-1, k)$

After Memoization

function binom(n, k)
  for $m \leftarrow 0, 1, \ldots, n$ **do**
    $b[m, 0] \leftarrow 1; \quad b[m, m] \leftarrow 1$
  for $m \leftarrow 2, 3, \ldots, n$ **do**
    for $\ell \leftarrow 1, 2, \ldots, m-1$ **do**
      $b[m, \ell] \leftarrow 0$
  return helper(n, k)

After Memoization (Continued)

function helper(m, ℓ)
  if $b[m, \ell] = 0$ **then**
    $b[m, \ell] \leftarrow$ helper$(m-1, \ell-1)$
        +helper$(m-1, \ell)$
  return $b[m, \ell]$

Trace binom(5,3)

Computing binomial coeffecients, 4

Building the Table Directly

function binom(n, k)
  for $m \leftarrow 0, 1, \ldots, n$ **do**
    $b[m, 0] \leftarrow 1; \quad b[m, m] \leftarrow 1$
  for $m = 2, 3, \ldots, n$ **do**
    for $\ell = 1, 2, \ldots, m-1$ **do**
      $b[m, \ell] \leftarrow b[m-1, \ell-1] + b[m-1, \ell]$
  return $b[n, k]$

Trace binom(5,3)

Computing binomial coeffecients, 5

Going from a recursion to a table-building computation.

Step 1. Give a recursive definition.
(For many problems, this is the hard part.)

Step 2. Memoize to exploit repeated subproblems.
(If there are few repeated subproblems, then memoization will not help.)

Step 3. Build the table directly to cut down overhead.
(If the answer depends on a small part of the table, then the recursion can be faster.)

Making Change—Again, 1

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

Example

- $d_1 = 1, d_2 = 4, d_3 = 6;\ a = 8$.
- The optimal choice is $\{ 4, 4 \}$.
- The greedy algorithm produces $\{ 6, 1, 1 \}$.

The Optimal Substructure of MCP

If: an optimal solution of the MCP for a uses a d_i-coin,
then: the rest of the coins give an optimal solution of the MCP for $a - d_i$.

(Why?)

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ \, mcn(a - d_i) \mid d_i \leq a \;\; \& \;\; 1 \leq i \leq k \, \}, & \text{if } a > 0. \end{cases}$$

Example

$$d_1 = 1$$
$$d_2 = 4$$
$$d_3 = 6$$

$$a = 8$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ \, mcn(a - d_i) \mid d_i \leq a \;\; \& \;\; 1 \leq i \leq k \, \}, & \text{if } a > 0. \end{cases}$$

Example

$$mcn(0) \quad = 0 \qquad\qquad = 0$$

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ \, mcn(a - d_i) \mid d_i \leq a \;\; \& \;\; 1 \leq i \leq k \, \}, & \text{if } a > 0. \end{cases}$$

Example

$$d_1 = 1$$
$$d_2 = 4$$
$$d_3 = 6$$

$$a = 8$$

$$mcn(0) = 0 \qquad = 0$$
$$mcn(1) = 1 + \min\{ \, mcn(0) \, \} \qquad = 1$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min\{\, mcn(a - d_i) \mid d_i \leq a \ \ \& \ \ 1 \leq i \leq k \,\}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned} mcn(0) &= 0 && = 0 \\ mcn(1) &= 1 + \min\{\, mcn(0) \,\} && = 1 \\ mcn(2) &= 1 + \min\{\, mcn(1) \,\} && = 2 \end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ mcn(a - d_i) \mid d_i \leq a \;\; \& \;\; 1 \leq i \leq k \}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned} mcn(0) &= 0 && = 0 \\ mcn(1) &= 1 + \min\{ mcn(0) \} && = 1 \\ mcn(2) &= 1 + \min\{ mcn(1) \} && = 2 \\ mcn(3) &= 1 + \min\{ mcn(2) \} && = 3 \end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min\left\{ mcn(a - d_i) \;\middle|\; d_i \leq a \;\;\&\;\; 1 \leq i \leq k \right\}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned}
mcn(0) &= 0 && = 0 \\
mcn(1) &= 1 + \min\{ mcn(0) \} && = 1 \\
mcn(2) &= 1 + \min\{ mcn(1) \} && = 2 \\
mcn(3) &= 1 + \min\{ mcn(2) \} && = 3 \\
mcn(4) &= 1 + \min\{ mcn(3), mcn(0) \} && = 1
\end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ mcn(a - d_i) \mid d_i \le a \ \& \ 1 \le i \le k \}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned}
mcn(0) &= 0 && = 0 \\
mcn(1) &= 1 + \min\{ mcn(0) \} && = 1 \\
mcn(2) &= 1 + \min\{ mcn(1) \} && = 2 \\
mcn(3) &= 1 + \min\{ mcn(2) \} && = 3 \\
mcn(4) &= 1 + \min\{ mcn(3), mcn(0) \} && = 1 \\
mcn(5) &= 1 + \min\{ mcn(4), mcn(1) \} && = 2
\end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ mcn(a - d_i) \mid d_i \leq a \ \& \ 1 \leq i \leq k \}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned}
mcn(0) &= 0 && = 0 \\
mcn(1) &= 1 + \min\{ mcn(0) \} && = 1 \\
mcn(2) &= 1 + \min\{ mcn(1) \} && = 2 \\
mcn(3) &= 1 + \min\{ mcn(2) \} && = 3 \\
mcn(4) &= 1 + \min\{ mcn(3), mcn(0) \} && = 1 \\
mcn(5) &= 1 + \min\{ mcn(4), mcn(1) \} && = 2 \\
mcn(6) &= 1 + \min\{ mcn(5), mcn(2), mcn(0) \} && = 1
\end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min \{ \, mcn(a - d_i) \mid d_i \leq a \;\; \& \;\; 1 \leq i \leq k \, \}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned}
mcn(0) &= 0 && = 0 \\
mcn(1) &= 1 + \min\{ \, mcn(0) \, \} && = 1 \\
mcn(2) &= 1 + \min\{ \, mcn(1) \, \} && = 2 \\
mcn(3) &= 1 + \min\{ \, mcn(2) \, \} && = 3 \\
mcn(4) &= 1 + \min\{ \, mcn(3), mcn(0) \, \} && = 1 \\
mcn(5) &= 1 + \min\{ \, mcn(4), mcn(1) \, \} && = 2 \\
mcn(6) &= 1 + \min\{ \, mcn(5), mcn(2), mcn(0) \, \} && = 1 \\
mcn(7) &= 1 + \min\{ \, mcn(6), mcn(3), mcn(1) \, \} && = 2
\end{aligned}$$

Making Change—Again, 2

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) \equiv \text{the number of coins in an optimal solution to MCP for } a$$

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min\{\, mcn(a - d_i) \mid d_i \le a \;\; \& \;\; 1 \le i \le k \,\}, & \text{if } a > 0. \end{cases}$$

Example

$d_1 = 1$

$d_2 = 4$

$d_3 = 6$

$a = 8$

$$\begin{aligned}
mcn(0) &= 0 && = 0 \\
mcn(1) &= 1 + \min\{\, mcn(0) \,\} && = 1 \\
mcn(2) &= 1 + \min\{\, mcn(1) \,\} && = 2 \\
mcn(3) &= 1 + \min\{\, mcn(2) \,\} && = 3 \\
mcn(4) &= 1 + \min\{\, mcn(3), mcn(0) \,\} && = 1 \\
mcn(5) &= 1 + \min\{\, mcn(4), mcn(1) \,\} && = 2 \\
mcn(6) &= 1 + \min\{\, mcn(5), mcn(2), mcn(0) \,\} && = 1 \\
mcn(7) &= 1 + \min\{\, mcn(6), mcn(3), mcn(1) \,\} && = 2 \\
mcn(8) &= 1 + \min\{\, mcn(7), mcn(4), mcn(2) \,\} && = 2
\end{aligned}$$

Making Change—Again, 3

The Making Change Problem (MCP)

Given: coin denominations $d_1 < d_2 < \cdots < d_k$ and an amount a.
Find: the smallest collection of coins that is worth amount a.

$$mcn(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1 + \min\{\, mcn(a - d_i) \mid d_i \le a \;\&\; 1 \le i \le k \,\}, & \text{if } a > 0. \end{cases}$$

```
function mcn(d_1,...,d_k; a)
    integer array num[0..a]
    num[0] ← 0
    for a' ← 1 to a do                          // Trace mcn(1,4,6;8)
        num[a'] ← ∞
        for i ← 1 to k do
            if d_i ≤ a' then num[a'] ← min(num[a'], 1 + num[a' − d_i])
    return num[a]
```

Making Change—Again, 4

$$mcn'(i,a) \equiv \begin{cases}\text{the number of coins in an optimal solution to MCP for } a \\ \text{using denominations } d_1,\ldots,d_i\end{cases}$$

$$mcn'(i,a) = \begin{cases} 0, & \text{if } a = 0; \\ +\infty, & \text{if } i = 0 \text{ and } a > 0; \\ mcn'(i-1,a), & \text{if } 0 < a < d_i; \\ \min\left(mcn'(i-1,a), 1 + mcn'(i,a-d_i)\right), & \text{otherwise} \end{cases}$$

```
function mcn'(d_1, ..., d_k; a)
    integer array num[0..k, 0..a]          // Goal: num[i,a'] = mcn(d_1,...,d_i;a')
    for a' ← 1 to a do num[0,a'] ← 0
    for i ← 1 to k do                      // Trace mcn'(1,4,6;8)
        num[i,0] ← 0
        for a' ← 1 to k do
            if d_i > a' then num[i,a'] ← num[i−1,a']
            else num[i,a'] ← min(num[i−1,a'], 1 + num[i,a' − d_i])
    return num[k,a]
```

Making Change—Again, 5

Reconstructing the Solution to the MCP

Given: $num[i,a'] = \begin{cases} \text{the min number of coins of denominations } d_1,\ldots,d_i \text{ need} \\ \text{to make change for amount } a' \text{ where } 0 \le i \le k \text{ and } 0 \le \\ a' \le a \end{cases}$

Find: What coins make up the optimal solution.

function reconstruct($d_1,\ldots,d_k$;a,$num[0..k,0..a]$)
    coins $\leftarrow$ the empty list
    $a' \leftarrow a$; $i \leftarrow k$
    while $a' > 0$ **do**
        if $(d_i \le a'$ & $num[i,a'] \ne num[i-1,a'])$
            then Add i to the *coins* list; $a' \leftarrow a' - d_i$
            else $i \leftarrow i - 1$
    return *coins*

Longest Increasing Subsequences, 1

Definition

Suppose $S = a_1, \ldots, a_n$ is a sequence of numbers.

(a) A *subsequence* of S is a sequence of numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

(b) Such a subsequence is *increasing* when $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$.

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers.
Find: An increasing subsequence of maximal length.

Example

For $S = 5, 2, 8, 6, 3, 6, 9, 7$; a longest increasing subsequence is: 2, 3, 6, 9.

Longest Increasing Subsequences, 2

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

Given $S = a_1, \ldots, a_n$, we can turn this into a graph problem as follows:

Let $V = \{1, \ldots, n\}$, $E = \{(i,j) \mid i < j \;\&\; a_i < a_j\}$, and $G = (V, E)$.
G is a dag. *(Why?)*

$\therefore$ a longest increasing sequence in S $\equiv$ a longest path in G.

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

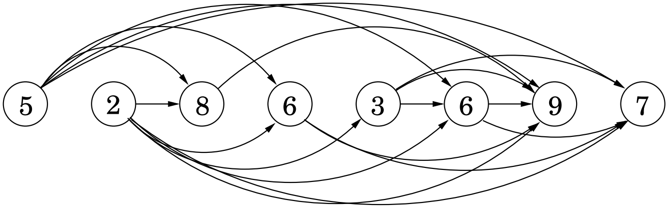


Image from DPV

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{(i,j) \mid i < j \;\&\; a_i < a_j\}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{L(i) \mid (i,j) \in E\}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example,* $L(1) = 1$.*)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1
a_i	5
$L(i)$	1
prev	0

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{(i, j) \mid i < j \ \& \ a_i < a_j\}$.

$$L(j) = \text{the length of a longest increasing subseq. ending at } j$$
$$= 1 + \max\{L(i) \mid (i, j) \in E\}.$$

Convention: $\max(\emptyset) = 0$. *(So, for example, $L(1) = 1$.)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2
a_i	5	2
$L(i)$	1	1
prev	0	0

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$
  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{(i, j) \mid i < j \ \&\ a_i < a_j\}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{L(i) \mid (i, j) \in E\}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example, $L(1) = 1$.)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3
a_i	5	2	8
$L(i)$	1	1	2
$prev$	0	0	1

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$

  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{ (i,j) \mid i < j \;\&\; a_i < a_j \}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{ L(i) \mid (i,j) \in E \}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example,* $L(1) = 1$.*)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3	4
a_i	5	2	8	6
$L(i)$	1	1	2	2
prev	0	0	1	1

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$
  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{ 1, \ldots, n \}$, $E = \{ (i,j) \mid i < j \;\&\; a_i < a_j \}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{ L(i) \mid (i,j) \in E \}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example,* $L(1) = 1$*.)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3	4	5
a_i	5	2	8	6	3
$L(i)$	1	1	2	2	2
$prev$	0	0	1	1	2

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$
  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{ (i, j) \mid i < j \ \& \ a_i < a_j \}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{ L(i) \mid (i, j) \in E \}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example,* $L(1) = 1$.*)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3	4	5	6
a_i	5	2	8	6	3	6
$L(i)$	1	1	2	2	2	3
prev	0	0	1	1	2	5

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$
  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{ (i,j) \mid i < j \;\&\; a_i < a_j \}$.

$$L(j) = \text{the length of a longest increasing subseq. ending at } j$$
$$= 1 + \max\{ L(i) \mid (i,j) \in E \}.$$

Convention: $\max(\emptyset) = 0$. *(So, for example, $L(1) = 1$.)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3	4	5	6	7
a_i	5	2	8	6	3	6	9
$L(i)$	1	1	2	2	2	3	4
$prev$	0	0	1	1	2	5	6

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$
  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Longest Increasing Subsequences, 3

Longest Increasing Subsequence Problem (LISP)

Given: A sequence of numbers. **Find:** A max-length increasing subsequence.

$G = (V, E)$, where $V = \{1, \ldots, n\}$, $E = \{(i, j) \mid i < j \;\&\; a_i < a_j\}$.

$$\begin{aligned} L(j) &= \text{the length of a longest increasing subseq. ending at } j \\ &= 1 + \max\{L(i) \mid (i, j) \in E\}. \end{aligned}$$

Convention: $\max(\emptyset) = 0$. *(So, for example, $L(1) = 1$.)*

Example (for $S = 5, 2, 8, 6, 3, 6, 9, 7$)

i	1	2	3	4	5	6	7	8
a_i	5	2	8	6	3	6	9	7
$L(i)$	1	1	2	2	2	3	4	4
$prev$	0	0	1	1	2	5	6	6

So the length of a LCS is 4 and the LCSs are:

- $a_2, a_5, a_6, a_8 = 2, 3, 6, 7$

  Note: $prev[8] = 6$, $prev[6] = 5$, $prev[5] = 2$, $prev[2] = 0$
- $a_2, a_5, a_6, a_7 = 2, 3, 6, 9$

Optimal Substructure

A problem has optimal substructure when an optimal solution is made up of optimal solutions to its subproblems.

Examples

(a) Shortest paths in a graph.
(b) Making change.
(c) ...

Non-examples

(a) Longest paths in a graph.
(b) 3SAT
(c) ...

Longest Common Subsequences, 1

Problem: Longest common subsequence

Given: Strings $s[1..m]$ and $t[1..n]$.
Find: The longest subsequence common to s and t.

Example

s = a b a z d c

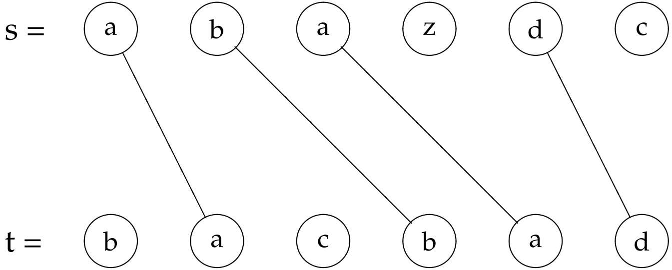
t = b a c b a d

Longest Common Subsequences, 1

Problem: Longest common subsequence

Given: Strings $s[1..m]$ and $t[1..n]$.
Find: The longest subsequence common to s and t.

Example



Credit: A. Blum, `http://www.cs.cmu.edu/~avrim/451f09/lectures/lect1001.pdf`

Longest Common Subsequences, 2

Simplification

Initially, we'll just worry about computing the *length* of the l.c.s.

Subproblems?

$$LCS[i, j] = \text{the length of the l.c.s. of } s[1..i] \text{ and } t[1..j].$$

Questions:

1. $LCS[m, n] = ??$
2. $LCS[0, j] = ??$
3. $LCS[i, 0] = ??$
4. $LCS[i, j] = ??$ (in terms of $LCS[i', j']$ for smaller i' and j')

Longest Common Subsequences, 3

$$LCS[i, j] = \text{the length of the l.c.s. of } s[1..i] \text{ and } t[1..j].$$

Questions:

1. $LCS[m, n]$ = the answer to the big problem

2 & 3 $LCS[0, j] = LCS[i, 0] = 0$.

4 $LCS[i, j] = ??$ (in terms of $LCS[i', j']$ for smaller i' and j') $(i, j > 0)$

Case: $s[i] = t[j]$. Then?

Case: $s[i] \neq t[j]$. Then?

Longest Common Subsequences, 4

$LCS[i,j]$ = the length of the l.c.s. of $s[1..i]$ and $t[1..j]$.

$$= \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ 1 + LCS[i-1, j-1], & \text{if } i, j > 0 \text{ and } s[i] = t[j]; \\ \max(LCS[i, j-1], LCS[i-1, j]) & \text{otherwise.} \end{cases}$$

Exercises for the reader

1. Build the table for $s = "abazdc"$ and $t = "bacdad"$.
2. Give the algorithm for computing $LCS[0..m, 0..n]$.
3. Given the table $LCS[0..m, 0..n]$ (and s and t),
   reconstruct the actual longest common subsequence.