❖ **Problem 1.** ***DPV Exercise 8.1.*** ❖ Suppose, given a matrix of distances M and a buget b, the procedure $S(M,d)$ returns a tour through all the cities that has length $\leq b$, if such a tour exists, and returns NONE, if there is no such tour. Consider:

function $TSP_{opt}(M)$
  $\ell \leftarrow 0$; $r \leftarrow$ the sum of all the numbers in M
  while $\ell < r$ **do**
    $m \leftarrow \lfloor(\ell+r)/2\rfloor$; $T \leftarrow S(M,m)$
    if $T =$ NONE **then** $\ell \leftarrow m+1$ **else** $r \leftarrow m$
  return $S(M,r)$

Let $s =$ the sum of all the numbers in M. Note: If $|M|$ = the number of bits it takes to write down M, then $|s|$ (= the number of bits it takes to write down s) is $O(|M|)$. Now, if $S(M,d)$ runs in $p(|M|,|d|)$-time for some polynomial p, then the above function on input M runs in $O(|M| \cdot p(|M|, c\cdot|M|))$ time, for some constant c.

❖ **Problem 2.** ***DPV Exercise 8.2.*** ❖ Suppose, given a graph G, the procedure $D(G)$ returns *true*, if G has a Rudrata path, and *false*, othewise. Here is how to use D to find a path if it exists.

function *RudrataPath*(G) *// where* $(V,E) = G$
  if not $D(G)$ **then return** *"no path"*
  $E' \leftarrow E$
  for each $e \in E$ **do**
    // See if we still have a Rudrata path when we leave out e
    $G' \leftarrow (V, E' - \{e\})$
    if $D(G')$ **then** $E' \leftarrow E' - \{e\}$
  return E'

The only edges left in E' at the end are the edges making up the Rudrata path—all the other edges could be left out.

❖ **Problem 3.** ***DPV Exercise 8.3.*** ❖ *We show: SAT* $\trianglelefteq$ *stingy SAT.*
Given an instance of SAT I, let (I,k) be an instance of stingy SAT where k = the number of variables in SAT instance I.
CLAIM: I is a yes-instance of SAT $\iff$ (I,k) is a yes-instance of stingy SAT.
PROOF:
($\Rightarrow$) Suppose I has a satisfying assignment $\mathscr{T}$. Since there are a total of k variables in I, $\mathscr{T}$ works as a positive solution for the stingy SAT instance (I,k) too.
($\Leftarrow$) Suppose (I,k) has a satisfying assignment $\mathscr{T}$ with no more than k variables assigned *true*. Then obviously $\mathscr{T}$ is a positive solution to I also.

❖ **Problem 4.** ***DPV Exercise 8.9.*** ❖ *We show: Vertex Cover* $\trianglelefteq$ *Hitting Set.*
Suppose $G = (V,E)$ and $b \in \mathbb{N}$ where $E = \{(u_1,v_1),\dots,(u_k,v_k)\}$. Let $f(G,b) = (\{S_1,\dots,S_k\},b)$ where for each i, $S_i = \{u_i,v_i\}$.
CLAIM: (G,b) is a yes-instance of Vertex Cover $\iff$ $(\{S_1,\dots,S_k\},b)$ is a yes-instance of Hitting Set.
PROOF:
($\Rightarrow$) Suppose there is a set $H \subseteq V$ with $|H| \leq b$ and each $e \in E$ has an end-point in H. Then $|H| \leq b$ and, for each i, $H \cap S_i \neq \emptyset$.
($\Leftarrow$) Suppose there is an $H \subseteq V$ with $|H| \leq b$ and, for each i, $H \cap S_i \neq \emptyset$. Then, by the definition of the S_i's, $|H| \leq b$ and each $e \in E$ has an end-point in H.

❖ **Problem 5.** ***DPV Exercise 8.10.*** ❖
- **(a)** Reduction from *Clique*: Given a graph G and $g > 0$, construct K_g = the complete graph on g vertices. Then (K_g, G) is an instance of subgraph isomorphism that has a positive answer iff G has a clique subgraph on g vertices.
- **(c)** Reduction from *SAT*: Given φ a conjunctive normal form formula, let $g =$ the number of clauses in φ. Then (φ, g) is an instance of *MAX SAT* that has a positive solution iff φ is satisfiable; and in fact, the same satisfying assignment works for both cases.
- **(d)** Reduction from *Clique:* Given a graph G and $g > 0$, let $a = g$ and $b = g\cdot(g-1)/2$. Then (K_b, a, b) is an instance of *Dense subgraph* that has a positive answer iff G has a clique subgraph on g vertices. (A clique on g vertices always has $g\cdot(g-1)/2$ edges.)